Sl No of QP : 5713
Unique Paper Code : 42354401
Name of the Paper : Real Analysis
Name of the Course : B.Sc. (Prog) Physical Sciences/Mathematical Sciences
Semester : IV
Duration : 3 Hours Maximum Marks: 75

(Write your Roll No. on the top immediately on receipt of this question paper.)
All questions are compulsory.
*Attempt any **two** parts from each question.*

1. (a) Define a countable set. Show that the set $\mathbb{Q}$ of rational numbers is countable.

(b) Define absolute value of a real number $'x'$.
Find all $x \in \mathbf{R}$ that satisfy the following inequalities:
(i) $4 < |x+2| + |x-1| < 5$
(ii) $|2x-1| \le x+1$

(c) (i) Define supremum of a non-empty bounded subset S of **R**.
(ii) Show that a real number u is the supremum of a non-empty subset S of **R** if and only if it satisfies the following conditions:
(1) $s \le u$ for all $s \in S$.
(2) For each positive real number ε, there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.
(6,6)

2. (a) State the Archimedean Property of real numbers. Show that if $x \in \mathbf{R}$, then there exists a unique $n \in \mathbf{Z}$ such that $n - 1 \le x < n$.

(b) Define the convergence of a sequence (x_n) of real numbers. Using the definition, evaluate the following limits:

(i) $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$
(ii) $\lim_{n\to\infty} \left(\frac{(-1)^n n}{n^2+1}\right)$

(c) Let (x_n) be a sequence of real numbers that converges to x and suppose that $x_n \ge 0$, $\forall n \in \mathbf{N}$. Show that the sequence $(\sqrt{x_n})$ converges to $\sqrt{x}$.

(6,6)

3. (a) Prove that every monotonically decreasing and bounded below sequence of real numbers converges.
(b) Show that the sequence (x_n) defined by
$x_1 = 1; x_{n+1} = \frac{1}{4}(2x_n + 3)$, $\forall n \ge 1$ is convergent. Also, find $\lim_{n\to\infty} x_n$.

(c) State Cauchy's Convergence Criterion for sequences of real numbers. Show directly from the definition that the following sequence is a Cauchy sequence:

$$\left(1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right).$$

(6.5, 6.5)

4. (a) State and prove Comparison test for positive term series. Hence, show that the following series converges:

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots$$

(b) Suppose that (x_n) is a sequence of non-negative real numbers. Prove that the series $\sum x_n$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded.

(c) (1) State (without proof) D'Alembert's ratio test for an infinite series.

(2) Test for convergence the series:

(i) $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+---.$

(ii) $\frac{1}{\log 2}+\frac{1}{(\log 3)^2}+\frac{1}{(\log 4)^3}+\cdots$

(6.5, 6.5)

5. (a) (i) Define an absolutely convergent series. Is every convergent series absolutely convergent? Justify your answer.

(ii) Test for convergence the series:

(1) $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}---$

(2) $\sum_{n=1}^{\infty}(-1)^n.e^{-n}$

(b) Show that if $a > 0$, then the sequence $\left(\frac{nx}{1+n^2x^2}\right)$ converges uniformly on the interval $[a,\infty)$ but not uniformly on the interval $[0,\infty)$.

(c) State Weierstrass M-Test for uniform convergence of series. Hence, show that

$$\sum\frac{1}{x^2+n^2},\quad \forall x\in \mathbf{R}$$

is uniformly convergent.

(6.5,6.5)

6. (a) Find the radius of convergence and exact interval of convergence of the power series

$$\sum\frac{n+1}{(n+2)(n+3)}x^n.$$

(b) Show that the function $f(x) = x^2$ defined on the interval $[0, b]$, where $b > 0$ is Riemann integrable.

(c) Show that every continuous function defined on $[a, b]$ is Riemann integrable.

(6,6)